

CLASSIFICATION OF RANK 5 PREMODULAR CATEGORIES

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ABSTRACT. We survey a number of classification tools developed in recent years and employ them to classify pseudo-unitary rank 5 premodular categories up to Grothendieck equivalence.

1. INTRODUCTION

Fusion categories axiomatize and generalize the theory of representation theory, and their study encompasses not only the representation of finite groups and Lie groups, but also Hopf algebras. In many of these situations the fusion category associated to the underlying group/algebra enjoys extra structure, such as a notion of commutativity (*braiding*), duality (*rigidity*), non-degeneracy (*modularity*), or other compatibility conditions such as a *spherical structure* [1]. The study of fusion categories has moved beyond its roots in groups and algebras and now have more widespread use. For instance, nondegenerate ribbon braided fusion categories, *i.e.*, modular categories, have broad uses in physics where they describe topological phases of matter, and topological quantum computers [20, 22]. Modular categories also have applications in pure mathematics providing knot, link, and 3-manifold invariants through TQFT [19]. More recently, ribbon braided fusion categories, *i.e.*, premodular categories, have garnered increased attention. These categories are thought to describe higher dimensional TQFT and thus have relevance in manifold invariants and physics [18]. Furthermore, the study of premodular and modular categories often follow a “leap-frogging” pattern whereby advances in the understanding of one type of category allow for advances in another. In recent years, researchers have found it useful to stratify fusion categories by a numeric parameter known as their *rank*. The low-rank classification of premodular categories lag the classification of modular categories. On the premodular side categories are understood through rank 4 [16, 17, 2], while on the modular side they are completely characterized through rank 5 and partially characterized through rank 11 [14, 6, 3, 21]. The lack of advancement in the classification of premodular categories is slowing the classification of modular categories. In this work we aim to utilize a technique known as *de-equivariantization* to produce modular categories from premodular categories. This will allow us to leverage recent advances in arithmetic properties of modular categories, *e.g.* [4], to classify pseudo-unitary premodular categories of rank 5 up to Grothendieck equivalence. Specifically, we will show:

Theorem 1.1. *If \mathcal{C} is a pseudo-unitary rank 5 premodular category, then*

- \mathcal{C} is symmetric and is given by $\text{Rep}(G, z)$ where G is \mathbb{Z}_5 , D_8 , Q_8 , D_{14} , $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, \mathfrak{S}_4 , or \mathfrak{A}_5 , and z is a central element of order at most 2.
- \mathcal{C} is properly premodular and Grothendieck equivalent to:
 - $\text{Rep}(D_{14})$ with $\mathcal{C}' \cong \text{Rep}(\mathbb{Z}_2)$, and $d_i = (1, 1, 2, 2, 2)$.
 - $\text{Rep}(\mathfrak{S}_4)$ with $\mathcal{C}' \cong \text{Rep}(\mathfrak{S}_3)$, $d_i = (1, 1, 2, 3, 3)$, and $T = \text{diag}(1, 1, 1, -1, -1)$.

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- $\text{Rep}(D_8)$ with $\mathcal{C}' \cong \text{Rep}(\mathbb{Z}_2)$, $d_i = (1, 1, 2, 1, 1)$, and $T = \text{diag}(1, 1, \theta, -1, -1)$, where θ is a root of unity satisfying a monic degree 4 polynomial over \mathbb{Z} .
- $\text{PSU}(2)_8$, and is obtainable as a \mathbb{Z}_2 -equivariantization of $\text{Fib} \boxtimes \text{Fib}$.
- \mathcal{C} is modular and it is Grothendieck equivalent to $SU(2)_4$, $SU(2)_9/\mathbb{Z}_2$, $SU(5)_1$, or $SU(3)_4/\mathbb{Z}_3$ [4].

Moreover, each case is realized.

In Section 2 we will review the basic theory of premodular categories. Having dispensed with the preliminaries we will stratify premodular categories by the amount of degeneracy. In Section 3 we will analyze each case in turn to arrive at Theorem 3.2.

2. PRELIMINARIES

A premodular category \mathcal{C} is a braided, balanced, and fusion category. We will denote the isomorphism classes of simple objects by X_a , indexed such that $X_0 = \mathbb{I}$ is the monoidal unit. We will denote the set of such isomorphism classes by $\text{Irr}(\mathcal{C})$. The fusion matrices, N_a , the S -matrix, $S = (s_{x,y})$, and the T -matrix, $T = (\delta_{x,y}\theta_x)$ are defined in the usual way [1]. Here θ_x is the twist of the simple x and is known to have finite order [15]. The triple $(\{N_a\}, S, T)$ is known as **premodular datum**. Throughout we will assume \mathcal{C} is **pseudo-unitary** and so we may take the categorical dimensions, d_a , are the Frobenius-Perron eigenvalues of the N_a , *i.e.*, the FP-dimensions.

Let \mathcal{C}' denote the Müger center of the category \mathcal{C} [12]. We say that \mathcal{C} is **modular** if $\mathcal{C}' = \text{Vec}$, **symmetric** if $\mathcal{C} = \mathcal{C}'$, and **properly premodular** otherwise. Symmetric categories are completely characterized by the following result due to Deligne:

Theorem 2.1 (See [7]). *If \mathcal{C} is a symmetric ribbon category, then $\mathcal{C} \cong \text{Rep}(G, z)$ where z is a central element of G of order at most 2.*

In the properly premodular setting, this result can be exploited to determine part of the premodular datum. For instance, the categorical dimensions of simples in $\mathcal{C}' \cong \text{Rep}(G, z)$ follow from the representation theory of G . Of course, by definition of the S -matrix, for x, y isomorphism classes of simples in \mathcal{C}' , one has $s_{x,y} = d_x d_y$. One can obtain similar information regarding the T -matrix, for instance:

Lemma 2.2. *Suppose \mathcal{C} is a symmetric category and $X \in \mathcal{C}$ is an element of order 2, then $\theta_X = \pm 1$. Moreover, $\theta_X = 1$ if X fixes an element of \mathcal{C} . In particular, this is true if \mathcal{C} is the Müger center of an odd rank premodular category.*

Proof. The first statement of this lemma is proven in [7]. Now if $\theta_X = -1$ and X is an element of order 2 we know that $\langle 1, X \rangle \cong \text{sVec}$. Now by [11, Proposition 2.6 (i)] we know that X moves everything in the category. \square

In the case that \mathcal{C} has odd rank, this lemma tells us that the Müger center of \mathcal{C} is Tannakian. In particular, we can exploit the minimal modularization of [23]. This procedure is more generally known as de-equivariantization. Recent work of Natale and Bruciu allow one to gain a great deal of insight into the structure of the minimal modularization of \mathcal{C} . It is often true that the structure enforced by the de-equivariantization procedure is at odds with the Galois theory of the modular category. A complete discussion of de-equivariantization and the Galois theory of

modular categories is beyond the scope of this current work, but further details can be found in [23, 5, 6].

Next, recall that the T -matrix, S -matrix, and dimensions are related through the **balancing equation** [1]:

$$\theta_x \theta_y S_{xy} = \sum_{k=0}^4 N_{x^*y}^k \theta_k d_k \quad (1)$$

Furthermore, recall that the d_x are further related through the **dimension equation**:

$$d_x d_y = \text{Tr}_C(X \otimes Y) = \text{Tr}_C\left(\bigoplus N_{x,y}^z Z\right) = \sum_z N_{x,y}^z d_z \quad (2)$$

The structure of the fusion rules, knowledge of dimensions, and twists of a subcategory often allow one to produce a pair of polynomials from these equations from which the order of certain twists can be bounded. In particular, we have:

Lemma 2.3. *If \mathbf{D} is a subset of $\{d_a \mid 1 \leq a \leq r\}$, $f \in \mathbb{Q}(\mathbf{D})[x]$ is a degree 2 polynomial with leading coefficient a_2 and constant coefficient a_0 , such that $f(\theta) = 0$ for some θ , then*

$$[\mathbb{Q}(\theta) : \mathbb{Q}] \leq \begin{cases} 2^{|\mathbf{D}|+1} & \text{if } a_2 \neq 0 \text{ and } a_0/a_2 \text{ is a unit} \\ 2^{|\mathbf{D}|} & \text{if } a_2 = 0 \text{ or } a_0/a_2 \text{ is not a unit.} \end{cases}$$

Proof. Let α be a primitive element of $\mathbb{Q}(\mathbf{D})$. Then the minimal polynomial of θ over $\mathbb{Q}(\mathbf{D})$ divides f in $\mathbb{Q}(\mathbf{D})$. In particular, if we let m denote the degree of $\mathbb{Q}(\mathbf{D}, \theta)$ over $\mathbb{Q}(\mathbf{D})$, then $m \leq 2$. Note that the minimal polynomial of θ over $\mathbb{Q}(\mathbf{D})$ must divide the minimal polynomial of θ over \mathbb{Q} . Since θ is a root of unity we can conclude that the minimal polynomial over $\mathbb{Q}(\mathbf{D})$ can only possibly have degree 2 if $a_2 \neq 0$ and a_0/a_2 is a unit. In particular,

$$m \leq \begin{cases} 2 & \text{if } a_2 \neq 0 \text{ and } a_0/a_2 \text{ is a unit} \\ 1 & \text{if } a_2 = 0 \text{ or } a_0/a_2 \text{ is not a unit.} \end{cases}$$

Next note that for any $d \in \mathbf{D}$, d satisfies a degree 2 monic polynomial over $\mathbb{Q}(\mathbf{D} \setminus \{d\})$. In particular, $\mathbb{Q}(\mathbf{D})$ has degree at most $2^{|\mathbf{D}|}$ over \mathbb{Q} . Thus $\mathbb{Q}(\mathbf{D}, \theta)$ has degree at most $m 2^{|\mathbf{D}|}$ over \mathbb{Q} . Since $\mathbb{Q}(\theta)$ is a subfield the result follows. \square

Lemma 2.4. *Let \mathcal{C} be a rank r premodular category with rank $r - 1$ Müger center, and order the simples so that the last object, X_r , is not in \mathcal{C}' . Then $S_{r,r} = -\dim \mathcal{C}'$.*

Proof. Since X_r is not in the Müger center, the r -th column of the S -matrix must be orthogonal to the 0-th column. However, $S_{\ell,0} = d_\ell$ for $1 \leq \ell \leq r$, and $S_{k,r} = d_k d_r$ for $1 \leq k \leq r - 1$. Thus the orthogonality condition reads $d_r \dim \mathcal{C} = d_r S_{r,r}$. \square

3. CLASSIFICATION OF RANK 5 PREMODULAR CATEGORIES

In this section we will classify rank 5 pseudo-unitary premodular categories upto premodular datum, and in particular Grothendieck equivalence. The classification of pseudo-unitary rank 5 modular categories can be found in [4] so it suffices to consider symmetric and properly premodular categories.

We begin with symmetric premodular categories. By Theorem 2.1 and Lemma 2.2 it suffices to determine all groups which have exactly 5 irreducible representations. The number of irreducible representations can be related to the order of the group via a classical number theoretic argument [3, lemma 4.4 (ii)]. In particular, we have

$$5 \leq |G| \leq A_5 \implies 5 \leq |G| \leq 1806$$

Applying these bounds one can perform an exhaustive search in GAP to deduce:

Proposition 3.1. *If \mathcal{C} is a symmetric rank 5 category, then \mathcal{C} is Tannakian and is given by $\text{Rep}(G)$ where G is \mathbb{Z}_5 , D_8 , Q_8 , D_{14} , $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, \mathfrak{S}_4 , or \mathfrak{A}_5 .*

Having completed the classification of rank 5 symmetric categories we find it convenient to stratify our analysis of rank 5 properly premodular categories by the rank of the Müger center. From Propositions 3.1, 3.3, 3.7, and 3.11 we will be able to prove:

Theorem 3.2. *If \mathcal{C} is a pseudo-unitary rank 5 premodular category, then*

- \mathcal{C} is symmetric and is given by $\text{Rep}(G, z)$ where G is \mathbb{Z}_5 , D_8 , Q_8 , D_{14} , $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, \mathfrak{S}_4 , or \mathfrak{A}_5 , and z is a central element of order at most 2.
- \mathcal{C} is properly premodular and Grothendieck equivalent to:
 - $\text{Rep}(D_8)$ with $\mathcal{C}' \cong \text{Rep}(\mathbb{Z}_2)$, $d_i = (1, 1, 2, 1, 1)$, and $T = \text{diag}(1, 1, \theta, -1, -1)$, where θ is a root of unity satisfying a monic degree 4 polynomial over \mathbb{Z} .
 - $\text{Rep}(D_{14})$ with $\mathcal{C}' \cong \text{Rep}(\mathbb{Z}_2)$, and $d_i = (1, 1, 2, 2, 2)$.
 - $\text{Rep}(\mathfrak{S}_4)$ with $\mathcal{C}' \cong \text{Rep}(\mathfrak{S}_3)$, $d_i = (1, 1, 2, 3, 3)$, and $T = \text{diag}(1, 1, 1, -1, -1)$.
 - $\text{PSU}(2)_8$, and is obtainable as a \mathbb{Z}_2 -equivariantization of $\text{Fib} \boxtimes \text{Fib}$.
- \mathcal{C} is modular and it is Grothendieck equivalent to $SU(2)_4$, $SU(2)_9/\mathbb{Z}_2$, $SU(5)_1$, or $SU(3)_4/\mathbb{Z}_3$ [4].

Moreover, each case is realized.

3.1. Rank 4 Müger Center.

Proposition 3.3. *There is no rank 5 properly premodular category with a rank 4 Müger center.*

In order to prove this result we proceed through Propositions 3.4, 3.5, 3.6. These propositions consider the different structures of \mathcal{C}' .

By Theorem 2.1, the structure of \mathcal{C}' is $\text{Rep}(G, z)$ as a symmetric category. Moreover, by [3, Lemma 4.4(ii)] we have $|G| \leq 42$. Using GAP we can conclude that G is isomorphic to \mathbb{Z}_4 , $\mathbb{Z}_2 \times \mathbb{Z}_2$, D_{10} , or \mathfrak{A}_4 . In each of these cases we may apply Lemma 2.4 to deduce that $s_{4,4} = -4$. Furthermore, in each case we can apply Theorem 2.1 and Lemma 2.2 along with (1) applied to $S_{a,a}$ for $0 \leq a \leq 4$ to deduce that $\theta_a = 1$ for $0 \leq a \leq 3$. In particular, the S - and T -matrices are determined by the dimension, d , and twist, θ , of X_4 . Moreover, by Lemma 2.3 we can conclude that the order, n , of θ satisfies $n \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$. The cases of \mathbb{Z}_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are sufficiently similar that we consider them in tandem.

Proposition 3.4. *There is no rank 5 premodular category with Müger center Grothendieck equivalent to $\text{Rep}(\mathbb{Z}_4)$ or $\text{Rep}(\mathbb{Z}_2^2)$.*

Proof. In both cases the only $n \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$ consistent with equations (1) and (2) is $n = 10$, consequently $d = 1 + \sqrt{5}$. Letting G denote the group such that $\mathcal{C}' \cong \text{Rep}(G)$ we can de-equivariantize to produce a modular category \mathcal{C}_G . Let \hat{D} denote the set of orbits of isomorphism classes of the G -action and let D denote a transversal of \hat{D} . Then the simples of

\mathcal{C} can be understood via equivariantization as pairs (y, V) where y runs over D and V ranges over the simple projective representations of $\text{Stab}_G(y)$. Since \mathbb{I} is stabilized by G and G is an abelian group, four of the simples in \mathcal{C} are of the form (\mathbb{I}, χ_a) where χ_a is a irreducible character of G . The remaining simple is (y, V) where V is the sole simple projective representation of $\text{Stab}_G(y)$. In the case of $G \cong \mathbb{Z}_4$ we recall that $H^2(\mathbb{Z}_{2k}, \mathbb{C}^\times) = 0$ for all k , and so we may take V to be the sole irreducible linear representation. In particular, $\dim(y) = (1 + \sqrt{5})/4 \notin \mathbb{Z}$, an impossibility. So it suffices to consider $G \cong \mathbb{Z}_2^2$. In this case $H^2(\mathbb{Z}_2^2, \mathbb{C}^\times) \cong \mathbb{Z}_2$ and so, V must be the 2-dimensional simple projective corresponding to the nontrivial 2-cocycle in $H^2(\mathbb{Z}_2^2, \mathbb{C}^\times)$. In particular, $\mathcal{C}_{\mathbb{Z}_2^2} \cong \text{Fib}$. This is not possible [27]. \square

Proposition 3.5. *There is no rank 5 premodular category \mathcal{C} with $\mathcal{C}' = \text{Rep}(D_{10})$.*

Proof. The only choice for $n \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$ consistent with equations (1) and (2) are $n = 4$ or 12 . In both cases we have $d = 3$ and $\dim \mathcal{C} = 19$. This is not possible as $\dim \mathcal{C}' = 10 \nmid \dim \mathcal{C} = 19$. \square

Proposition 3.6. *There is no rank 5 premodular category \mathcal{C} with $\mathcal{C}' = \text{Rep}(A_4)$.*

Proof. The only choice for $n \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$ consistent with equations (1) and (2) are $n = 4$ or 12 . In both cases we have $d = 2\sqrt{3}$. Next we observe that $\mathbb{Z}_3 \subset \mathfrak{A}_4$ and so $\mathcal{C}_{\mathfrak{A}_4}$ is a premodular category with \mathbb{Z}_3 action. Letting D denote a transversal of the orbits of the simple object isomorphism classes under this action we can conclude that the objects in \mathcal{C} are pairs (y, V) where $y \in D$ and V is a simple projective representation of $\text{Stab}_{\mathbb{Z}_3}(y)$. Since $y = \mathbb{I}$ is fixed by the \mathbb{Z}_3 -action we have can deduce that three of the simples are (\mathbb{I}, χ_a) where the χ_a are the irreducible characters of \mathbb{Z}_3 . The remaining two objects we denote by (y, V) and (z, W) . Since \mathbb{Z}_3 has no proper subgroups it must be that $y \neq z$ and $\text{Stab}_{\mathbb{Z}_3}(y) = \text{Stab}_{\mathbb{Z}_3}(z) = 1$. Applying the orbit-stabilizer theorem we know that $|\mathbb{Z}_3 \cdot y| = |\mathbb{Z}_3 \cdot z| = 3$. Thus y has dimension $2/\sqrt{3} \notin \mathbb{Z}$, an impossibility. \square

3.2. Rank 3 Müger Center.

Proposition 3.7. *If \mathcal{C} is a rank 5 properly premodular category with a rank 3 Müger center, then \mathcal{C} is Grothendieck equivalent to $\text{Rep}(\mathfrak{S}_4)$, $\mathcal{C}' \cong \text{Rep}(\mathfrak{S}_3)$, and $T = \text{diag}(1, 1, 1, -1, -1)$.*

We will consider the different possibilities for \mathcal{C}' in Propositions 3.9 and 3.10. However, before doing this we find it useful to establish the following lemma.

Lemma 3.8. *Suppose \mathcal{C} is a modular category with $d_a = d_b \neq 1$ for $a \neq b$ and there exists $\sigma \in \text{Gal}(\mathcal{C})$ such that $\sigma(a) = 0$, then $d_{\sigma(b)} = \pm 1$. In particular, $\mathcal{C}_{pt} \not\cong \text{Vec}$.*

Proof. The result follows by applying [14, Theorem 2.7], and in particular $d_{\sigma(0)}\sigma(d_k) = d_{\sigma(k)}\epsilon_{\sigma(k), \sigma}$, to $\sigma(d_a) = \sigma(d_b)$. \square

Next we apply [3, Lemma 4.4(ii)] to conclude that groups with 3 irreducible representations must satisfy $3 \leq |G| \leq A_3 \implies 3 \leq |G| \leq 6$. In particular, $G \cong \mathfrak{S}_3$ or \mathbb{Z}_3 .

Proposition 3.9. *There is no rank 5 properly premodular category whose Müger center is Grothendieck equivalent to $\text{Rep}(\mathbb{Z}_3)$.*

Proof. Suppose \mathcal{C} is such a properly premodular category. Then \mathcal{C}' is Tannakian since \mathbb{Z}_3 has no order 2 central elements. Thus $\mathcal{C}_{\mathbb{Z}_3}$ is modular. Now let \hat{D} be the set of orbits of isomorphism classes of simples in $\mathcal{C}_{\mathbb{Z}_3}$ under the \mathbb{Z}_3 action, and let D be a transversal of \hat{D} . Then the simples

in \mathcal{C} are (\mathbb{I}, χ_a) , (y, V) , and (z, W) , where χ_a are the irreducible linear characters of \mathbb{Z}_3 , and V and W are simple projective representations of $\text{Stab}_{\mathbb{Z}_3}(y)$ and $\text{Stab}_{\mathbb{Z}_3}(z)$ respectively. Since \mathbb{Z}_3 has no proper subgroups and $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$ we may take V and W to be irreducible linear representations. Of course all irreducibles of the stabilizer must appear and so $y \neq z$ and the stabilizers are trivial. Applying the Orbit-Stabilizer Theorem, we can conclude that $|\mathbb{Z}_3.y| = |\mathbb{Z}_3.z| = 3$ and $\mathcal{C}_{\mathbb{Z}_3}$ is a rank 7 modular category of global dimension $1 + 3a^2 + 3b^2$ where $a = d_3/3$ and $b = d_4/3$. So either $a = b = 1$ by [6] or we may apply Lemma 3.8 to conclude that $a = 1$. Applying the equidimensionality of the universal grading of $\mathcal{C}_{\mathbb{Z}_3}$, [24, 8] we can deduce that $b = 1$ or 2 . The later case cannot occur as every integral modular category of rank 7 is pointed [6, Theorem 5.8]. Thus the global dimension of \mathcal{C} is 21. Applying [9, Theorem 6.3] we can conclude that \mathcal{C} is Grothendieck equivalent to $\text{Rep}(\mathbb{Z}_3 \rtimes \mathbb{Z}_7)$. Applying these fusion rules, balancing, and the symmetry $s_{34} = s_{43}$ allows us to conclude that $\theta_3 = \theta_4$. The last column of S is now determined in terms of θ_3 by balancing and the fusion rules. Applying orthogonality of the first and last column of the S -matrix produces the equation $\theta_3^2 + 5\theta_3 + 1 = 0$. This is not possible as θ_3 is a root of unity. \square

Proposition 3.10. *If \mathcal{C} is properly premodular and \mathcal{C}' has rank 3, then \mathcal{C} is Grothendieck equivalent to $\text{Rep}(\mathfrak{S}_4)$, $\mathcal{C}' \cong \text{Rep}(\mathfrak{S}_3)$, and $T = \text{diag}(1, 1, 1, -1, -1)$. Such a category can be obtained via the \mathfrak{S}_3 -equivariantization of a 3-fermion theory.*

Proof. Just as in the proof of Proposition 3.9 we may de-equivariantize and understand the simples of \mathcal{C} in terms of equivariantization. Since $H^2(\mathfrak{S}_3, \mathbb{C}^\times) = 0$ we have that the simples are (\mathbb{I}, ρ_a) , (y, V) , and (z, W) where ρ_a are the irreducible representations of \mathfrak{S}_3 , while V and W are the irreducible linear representations of $\text{Stab}_{\mathfrak{S}_3}(y)$ and $\text{Stab}_{\mathfrak{S}_3}(z)$ respectively. There are now two cases to consider:

Case 1: $y = z$ In this case $\text{Stab}_{\mathfrak{S}_3}(y)$ can only have two irreducible linear representations and hence is isomorphic to \mathbb{Z}_2 . By the Orbit-Stabilizer Theorem we can conclude that $\mathcal{C}_{\mathfrak{S}_3}$ has rank 4 and thus is pointed by [14].

By [5] we know that the simple objects in \mathcal{C} have dimensions 1, 1, 2, 3, 3 and the balancing equation easily tell us that $\theta_0 = \theta_1 = \theta_2 = 1$. In addition, if we de-equivariantize by $\text{Rep}(\mathbb{Z}_2) \subset \text{Rep}(\mathfrak{S}_3) = \mathcal{C}'$, we get a premodular category with simples with dimensions 1, 1, 1, 3. If we apply [5, Proposition 3.21] to the object of dimension 3 we can conclude that $x_3 = x_3^*$ and $x_4 = x_4^*$. In addition, we know that $X_1 \otimes X_3 = X_4 \implies X_3 \otimes X_3$ does not contain X_1 in its decomposition. If we now look at the dimension equation corresponding to $X_3 \otimes X_3$ we get the following Diophantine equation,

$$9 = 1 + 2N_{33}^2 + 3(N_{33}^3 + N_{33}^4) \implies N_{33}^2 = 1 \text{ and } N_{33}^3 + N_{33}^4 = 2 \text{ or } N_{33}^2 = 4 \text{ and } N_{33}^3 + N_{33}^4 = 0$$

The latter case cannot happen since this would force $4 \leq \|N_3\|_{\max} \leq d_3 = 3$ [21, Lemma 3.14]. From here we can finally conclude, $N_{23}^3 = N_{23}^4 = N_{24}^3 = N_{24}^4 = N_{13}^4 = 1$ and $N_{33}^3 + N_{33}^4 = N_{44}^3 + N_{44}^4 = 2$. If we now use the fact that all the N_i matrices commute, we can conclude that $N_{33}^4 = N_{44}^4$ and $N_{44}^3 = N_{33}^3$. If we combine the previous mentioned observations and the equation that you get by looking at s_{33} we are able to conclude that $\theta_3 = \theta_4 = -1$.

Note that if we consider the rank 4 pointed modular categories, \mathcal{D} , then the Grothendieck group is either \mathbb{Z}_4 , or \mathbb{Z}_2^2 . In the former case there are two \mathfrak{S}_3 equivariantizations of rank 9 and 12. In the later case, if we take \mathcal{D} to be equivalent to the toric code, then there are again two \mathfrak{S}_3 equivariantizations of rank 9 and 12. The final option is that \mathcal{D} is Grothendieck equivalent to the 3-fermion theory. In this case $\text{Aut}_{\otimes}^{\text{br}}(\mathcal{D}) \cong \mathfrak{S}_3$ and so the possible \mathfrak{S}_3 -actions are given by group homomorphisms, $\text{Hom}(\mathfrak{S}_3, \mathfrak{S}_3)$. Once again, there are rank 9 and 12, equivariantizations. However, the identity automorphism yields an \mathfrak{S}_3 -action on \mathcal{D} with 2-orbits. The stabilizer of

the nontrivial orbit is \mathbb{Z}_3 and so the \mathfrak{S}_3 -equivariantization under such an action would have rank 5 and have simples of dimension 1, 1, 2, 3, 3. Thus \mathcal{C} can be realized as an \mathfrak{S}_3 -equivariantization of the 3-fermion theory.

Case 2: $y \neq z$ Since all irreducibles must appear we have that the stabilizers of y and z in \mathfrak{S}_3 are trivial. Applying the Orbit-Stabilizer Theorem, we can conclude that $\mathcal{C}_{\mathfrak{S}_3}$ is a rank 13 modular category of dimension $1 + 6a^2 + 6b^2$ where $a = d_3/6$ and $b = d_4/6$. In the case that $\mathcal{C}_{\mathfrak{S}_3}$ is integral we may apply the techniques of [3] and by exhaustive search conclude that $a = b = 1$. On the other hand, if $\mathcal{C}_{\mathfrak{S}_3}$ is not integral in which case we may apply Lemma 3.8 to conclude that $a = 1$. Invoking the universal grading of [24] we have $b = 1$ or $b = \sqrt{7}$. In the later case $\mathcal{C}_{\mathfrak{S}_3}$ has dimension 49 contradicting [10, Proposition 8.32]. Thus $\mathcal{C}_{\mathfrak{S}_3}$ is pointed.

To see that this is not possible, let \mathcal{D} be a rank 13 pointed modular category. Of course, the orthogonal group on \mathbb{Z}_{13} , with quadratic form q coming from the twist on \mathcal{D} , $O(\mathbb{Z}_{13}, q)$, is either trivial or isomorphic to \mathbb{Z}_2 . So the actions of \mathfrak{S}_3 on \mathcal{D} are given by elements of $\text{Hom}(\mathfrak{S}_3, O(\mathbb{Z}_{13}, q))$, which are either trivial or the sign action. In the case of the trivial action, the isomorphism classes of simple objects in \mathcal{D} are fixed under the action of \mathfrak{S}_3 . In particular, the de-equivariantization has rank 39. In the case of the sign action, there are 7 orbits of simples under the \mathfrak{S}_3 action and hence the de-equivariantization has rank 21. Thus, \mathcal{C} is not the \mathfrak{S}_3 -equivariantization of a rank 13 pointed modular category.

□

3.3. Rank 2 Müger Center.

Proposition 3.11. *If \mathcal{C} is a rank 5 properly premodular category with a rank 2 Müger center, then one of the following is true:*

- (i) \mathcal{C} is Grothendieck equivalent to $\text{Rep}(D_8)$ with $\mathcal{C}' \cong \text{Rep}(\mathbb{Z}_2)$. Furthermore, the T -matrix is of the form $T = \text{diag}(1, 1, \theta, -1, -1)$ for some root of unity θ , corresponding to that 2-dimensional simple object, satisfying a monic quintic polynomial over \mathbb{Z} , or
- (ii) \mathcal{C} is Grothendieck equivalent to $\text{Rep}(D_{14})$.
- (iii) \mathcal{C} is Grothendieck equivalent to $\text{PSU}(2)_8$.

Moreover, all cases are realized. The first case by a \mathbb{Z}_2 -equivariantization of the toric code, the second by the adjoint subcategory of a 56-dimensional metaplectic category, e.g., $(\text{SO}(14)_2)_{ad}$, and the final case by a \mathbb{Z}_2 -equivariantization of $\text{Fib} \boxtimes \text{Fib}$.

Proof. Applying [3, Lemma 4.4(ii)] we see that if a group G has only two irreducible representations, then $G \cong \mathbb{Z}_2$. Thus \mathcal{C}' is Grothendieck equivalent to $\text{Rep}(\mathbb{Z}_2)$. Since \mathcal{C} has odd rank we can further conclude that \mathcal{C}' is Tannakian. Ordering the simples so that $X_0 = \mathbb{I}$ and X_1 generates \mathcal{C}' we have two cases to consider.

Case 1: $X_1 \otimes X_3 = X_4$ and $X_1 \otimes X_2 = X_2$. It follows immediately from these fusion rules that X_2 is self-dual. Next note that the de-equivariantization, $\mathcal{C}_{\mathbb{Z}_2}$ is a modular category of rank 4 and two of the objects have the same dimension. By [14] we can conclude that $\mathcal{C}_{\mathbb{Z}_2}$ is pointed, $\text{Fib} \boxtimes \text{Fib}$, or $\text{Fib} \boxtimes \text{Sem}$.

In the pointed case, we get $d_2 = 2$ and $d_3 = d_4 = 1$. Thus $X_2 \otimes X_2 = \mathbb{I} \oplus X_1 \oplus X_1 \oplus X_2$ or $\mathbb{I} \oplus X_1 \oplus X_3 \oplus X_4$. In the former case, X_2 would generate a 6-dimensional fusion subcategory. This is not possible as $6 \nmid 8$. Thus X_2 generates \mathcal{C} . Next note that this category is necessarily near group and hence can only be braided if $X_3^* = X_4$ [25], see also [13, Remark 4.4]. It follows

from [13, Theorem 4.2] that \mathcal{C} is Grothendieck equivalent to $\text{Rep}(D_8)$. Examining the balancing equation for $S_{1,3}$ we see that $\theta_3 = \theta_4$. Since X_2 is self-dual we can conclude that column 2 in S is real. In particular, $\theta_3 = \pm 1$. Since X_3 is not in the Müger center of \mathcal{C} we know that the 3-rd and 0-th column of S must be orthogonal, hence $\theta_3 = \theta_4 = -1$. By [2, Corollary 3.3] we know that

$$\theta^2 + \theta^{-2} = \frac{1}{D^2} \sum_{b,c} N_{b,c}^2 d_b d_c \left(\frac{\theta_b}{\theta_c} \right)^2 \in \mathbb{Z} \quad (3)$$

This yields a quintic polynomial for θ over \mathbb{Z} . Considering all possible n such that $\phi(n) < 4$ gives possible primitive roots. However, such $\theta = \zeta_n$ with $\phi(n) < 4$ satisfy (3). Such a category can be constructed by considering the equivariantization of the toric code under the nontrivial \mathbb{Z}_2 -action.

In the case where $\mathcal{C}_{\mathbb{Z}_2} = \text{Fib} \boxtimes \text{Fib}$, we get the following set of dimensions $\{1, 1, \varphi^2, \varphi^2, 2\varphi\}$ for \mathcal{C} , where φ is the golden mean. After computing the fusion rules using (1), we can conclude that \mathcal{C} is Grothendieck equivalent $PSU(2)_8$.

Finally, if $\mathcal{C}_{\mathbb{Z}_2} = \text{Fib} \boxtimes \text{Sem}$, we get a category \mathcal{C} with dimensions $\{1, 1, 1, 1, 2\varphi\}$. This category violates the condition given in [25, Theorem 1.1] for near-group categories, hence it does not exist.

Case 2: $X_1 \otimes X_a = X_a$ for $2 \leq a \leq 4$ In this case the de-equivariantization is rank 7 modular with simples of dimension $1, \hat{d}_1, \hat{d}_1, \hat{d}_2, \hat{d}_2, \hat{d}_3, \hat{d}_3$. In particular, by Lemma 3.8 and [6] we know that $(\mathcal{C}_{\mathbb{Z}_2})_{pt}$ is either $\text{Rep}(\mathbb{Z}_3)$ or $\mathcal{C}_{\mathbb{Z}_2}$. In the former case, applying the universal grading and the pigeon-hole principle gives $\hat{d}_2^2 + \hat{d}_3^2 = 3$. From here notice that the global dimension of this category is 9 and by [10, Proposition 8.32] we can conclude that $(\mathcal{C}_{\mathbb{Z}_2})_{pt} = \mathcal{C}_{\mathbb{Z}_2} \not\cong \text{Rep}(\mathbb{Z}_3)$, a contradiction.

Thus $\mathcal{C}_{\mathbb{Z}_2}$ is pointed and hence $d_1 = d_2 = d_3 = 2$. By the pigeon-hole principle and without loss of generality, we may assume X_2 is self-dual. Just as in case 1 a dimension argument shows that X_2 cannot be a subobject of $X_2 \otimes X_2$. So without loss of generality we have $X_2 \otimes X_2 = \mathbb{I} \oplus X_1 \oplus X_3$. Of course, the category generated by X_2 cannot exclude X_4 since its dimension must divide 14. Thus \mathcal{C} is cyclically generated by X_2 . Furthermore, since X_2 is self-dual, its tensor square must be as well. In particular, X_3 is self-dual. The pigeon-hole principle reveals that X_4 must also be self-dual. Thus \mathcal{C} is Grothendieck equivalent to $\text{Rep}(D_{14})$ by [13, Theorem 4.2].¹ Such categories may be realized as the adjoint categories of 56-dimensional metaplectic categories [26]. \square

Remark 3.12. *It is interesting to note that none of the known algebraic conditions, [2], for premodular categories are sufficient to determine θ in Proposition 3.11. Of course, there are finitely many modularizable premodular categories and so there must be finitely many such θ [21].*

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¹One can arrive at the same conclusion by consider equivariantizations of the pointed modular categories considered in [14] as in the proof of Proposition 3.10.

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